KNOTS
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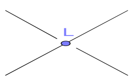
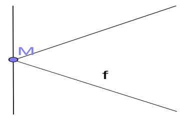
18.1 EXERCISE F)

Let $p: R^3 \to R^2$ be the natural projection $R^3 \to R^2$ given by $p(x_1, x_2, x_3) = (x_1, x_2)$. A crossing point in the projection of the knot K is a point $x \in R^2$ such that $p^{-1}(x) \cap K$ has two or more points. If $p^{-1}(x) \cap K$ is formed by two points, x is called a double point. In this case, if we can move slightly the knot so that the double point disappears we say that x is an unproper double point. Véase figura a continuación.



punto de cruce.



punto doble impropio.

Let's consider knots such that every crossing point is a proper double crossing point. Find all knots with one, two, three, four, five or six crossing points.

Answer:

The difficulty in this problem lies on the fact that when we regard for the first time the projection of a knot, the feeling we have is that the crossing points may appear disorganized in a caothic way.

The idea to solve this problem consists of organizing the proper double points appearence by observing that we always can consider the proper double points (to which may be reduced the crossing points) distributed in different rows in a fixed direction (this direction may be arbitrary).

Then the infinitude of possibilities of the aspect of the double points are restricted to the finite distributions of the finite number of the crossing points in rows. (See $Fig.1$, $Fig, 2$).

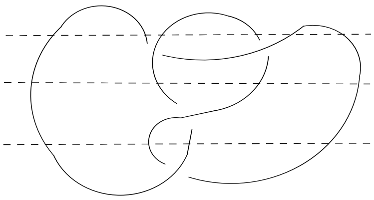


fig.1

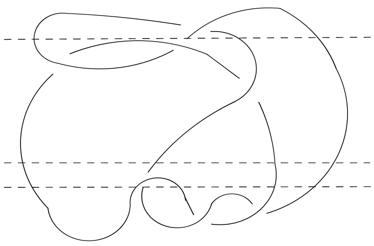


fig.2

When we unmake the inverse image of the projection of a cross so that the knot remains unchanged, we undo a cross or we obtain another cross which appears in a different row. In both cases of our distribution, is diminishing in one the crossing number points in a row and increasing it in another row (which could be empty at the beginning).

But when we consider from the start all possibilities in the distribution of crossing points in rows, this new situation cannot appear.

We get a one-dimension manifold by glueing the ends of the crosses in an appropiate way that we'll see later.

Arranged in rows the crosses, we can slide them in every row towards the left without changing the one-dimension manifold obtained neither the way of glueing the free ends of the crosses.

This is why we assume that crosses'rows give raise to not increasing length cross'columns.

Once arranged in rows the crosses, all in the third square of the plane with axis OX in OY; as the crosses give the same knot that the symmetrical crosses respect to the bisector of this square, we can assume that column's number is less than row‘s number.

In order to get the knot with the given crosses we have to unite the ends whithout doing more crosses.

Every cross has four ends. The crosses stay distributed also in horizontal rows (the double of the number of rows). The ends must be united by arcs not intersecting the crosses in any row. The exterior top ends in the first row must be united between them. The exterior botton ends are united between them and the sane happens with extreme ends on the right and on the left. Even more, lower ends in the inside of crosses in one row are united to the upper inner ends of crosses in the inmediate lower row.

If we unite the free ends of the same cross two by two the cross can be undone in the knot. This is why we assume that this is not the case.

As a consequence, in order to see how many knots are there with a determined number of double crossing points, we are going to consider every possibility in distributing the crosses in the rows and then we'll see the different possibilities arosen when we unite the free ends according to the rules already stated.

We have to consider two types of crosses:

In the more general case, when we unite the free ends in the crosses we get a closed 1-dimensional variety, independently of the type of cross. This 1-dimensional manifold is not a knot if it is not connected. This is why we study first the connection of the obtained manifold with only one type of crosses and then in the case that this manifold is connected the different types of knots we may construct changing the type of the crosses.

We'll also take into account that when we change in a knot every cross of one type into a cross of different type we obtain a knot image of the first knot on an horizontal mirror placed above the first knot, without touching it. If both knots can be taken one into the other, the knot is called amphichaeiral.

1) Let's assume that the knot has only a double point.

Then we must unite the upper free ends of the unique cross and the lower free ends between them being the cross unmade. Or we unite the two left free ends of the unique cross and the right free ends, being also the cross unmade. So the only knot with a only double point is TRIVIAL.

2) Let's assume that the knot has two double points.

We can distribuir them in one row or in one column, but both are equivalent.

Fig. 3

The only possibilities for uniting the free ends according to the rules and not producing more crosses are the drawn in picture 3. The first one gives rise to two connected component (this is not a knot) and the second one gives raise to the trivial knot. A change of cross type in just one cross keeps it trivial. The symmetrical knot respect to a mirror situated above of the trivial knot is trivial: THE ONLY KNOT WITH TWO DOUBLE POINTS IS THE TRIVIAL KNOT.

3) Let's assume the knot having three double points.

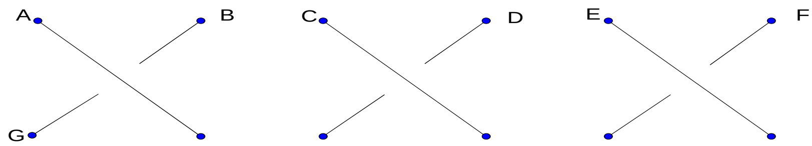


Fig. 4

We can distribute them in one row or in two rows. (We denote by capital letters the ends we are going to work with, see Fig.4.)

If we unite A to B or to G one cross will be undone. If we unite A with C we cannot unite B to another end in the same row whitout producing a new double point. If we unite A with D , in order not to have a new double point we have to unite E with F and then a cross will be undone. If we unite A with E; in order not to have none of the ends B,C,D free, it is necessary to create a new double point. Then we only can unite A to F.

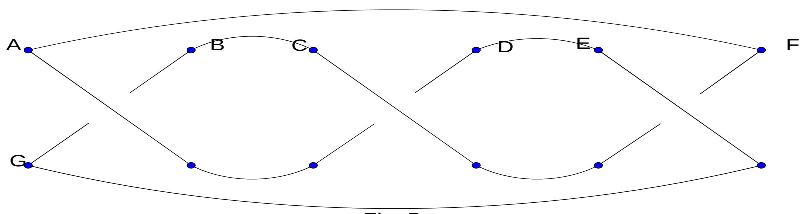


Fig. 5

We don't unite C and D because one cross will be undone and in order not to create more double points we unite B and C, also D and E.

The same happens to the lower free ends and we get a not trivial knot called trefoil knot.

If we change the type of one double point, the obtained knot is trivial. If we change the type of two double points we get the symmetrical of this last one, which is also the trivial knot. If we change the type of the three double points we get the symmetrical of the trefoil knot respect to the mirror, which is not equivalent to the trefoil knot. (It is seen in Ex. 18.7 C).

Now we put the double points in two rows: Fig. 6 a) and Fig. 6 b)

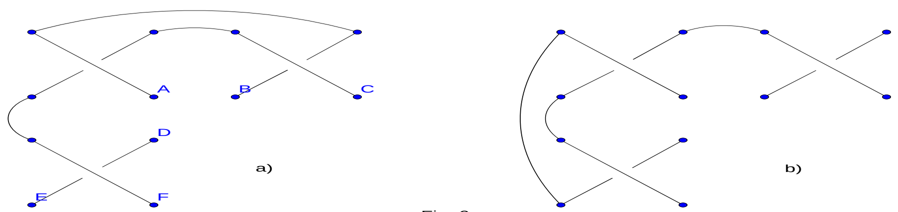
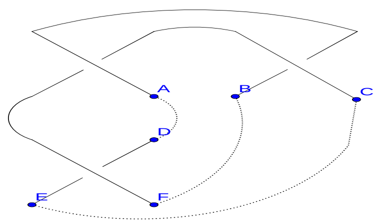


Fig. 6

There are two possibilities for uniting the upper free ends in the first row: both given on the right and in the left of Fig. 6.

Let's see case a):

Fig.7


We cannot unite A with C because then B would stay free and one more cross vould appear. The same happens if we unite A with F respect to D. If we unite A with E we should unite D with F so one cross would be undone.

We unite A with B or with D.

If we unite A with B we get more than one connected componnent.

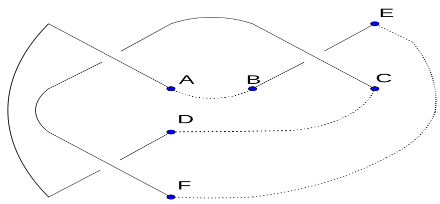
If we unite A with D the only thing we can do in order not to unmake crosses neither make more double crossing points is unite B with F and C with E.

The knot obtained is again the trefoil knot (See Fig. 7)

The different knots obtained changing the double points type are the same as the obtained in the previous case. (Three double points in a row).

Let's see case b):

Fig. 8



We cannot unite A with C neither with F because free ends remaine free. If we unite A with E we have to unite B with C and then we unmake a cross.

More than one connected component is produced uniting A with D. Then we only thing we can dois to unite A with B, then C with D and E with F, and we get again the trefoil knot of Fig. 8.

The different knots obtained changing type of crosses in this knot have alraedy been studied.

NOT TRIVIAL KNOTS WITH THREE DOUBLE POINTS ONLY ARE THE TREFOIL KNOT AND ITS SYMMETRICAL RESPECT TO THE MIRROR.

4) Let's assume the knot has four double points.

We can organize them in one, two or three rows.

a) On one row we have Fig. 9.

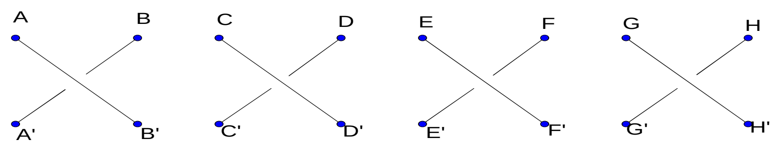


Fig. 9

When in fig. 9 we unite A with A' or H with H' we unmake one cross and we go to the previos case. The same happens if we unite A with B or G with H.

We cannot unite A with C, E or G because we had to produce more double points in order not to have free ends. We cannot either unite A with F because then we had to unite G with H and one cross will be undone. Only two possibilities remain: uniting A with D and A with H.

i) We unite A with D. Fig. 10

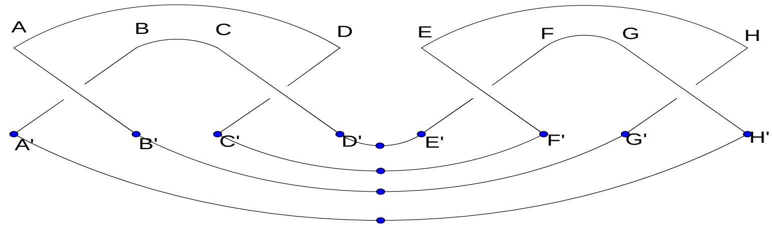


Fig10

Analogous reasonning with the lower free ends says that we only can unite A’ with D’ or with H’.

If we unite A’ with D’ we get more than one connected component, so we have to unite A’ with H’.

We cannot unite B’ with C’ because then more than one connected component will be produced, neither B’ with D’ because C’ would remain free.

When we unite B’ with E’ we have to unite F’ with G’ and we would have more than one connected component. Unitig C’ with D’ one cross will be undone. When we unite B’ with F’, G’ remains free, then the only possibility is to unite B’ with G’. As also when we unite C’ with E’, the end F’ remains free, we have to unite C’ with F’ and D’ with E’ obtaining two connected components,

So i) doesn’t give any knot.

ii) Let’ unite A with H. (Fig. 11 and Fig. 11‘)

When we unite B with C, in order not to undo crosses on the upper part, we have to unite D with E and F with G.

In this situation, when we unite A’ with B’ we undo one cross, the same happens if we unite A’ with F’, because there would be to unite G’ with H’. When we unite A’ with D’ more than one connected component will be produced. The only remaining possibility which doesn’t remain free ends is uniting A’ with H’. If now we unite B’ with G’, more than one connected component is produced and if we unite B’ with E’, there would be to unite C’ with D’, being unmade one cross. The only remained possibility with no free ends is to unite B’ with with C’ and now, in order not to undoing crosses is D’ with E’ and F’ with G’, but two connected components are produced. (Fig. 11).

When we unite B with E one cross in undone, and if we unite B with G, in order to not to have undone crosses we unite C with F and D with E. Now, in order to not to obtain the symmetrical of the previous i), (Fig. 10) we have to unite A' with H', B' with G', C' with F' and D' with E', but in this way we get more than two connected components.

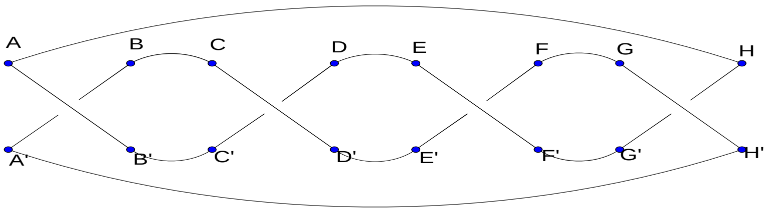


Fig 11

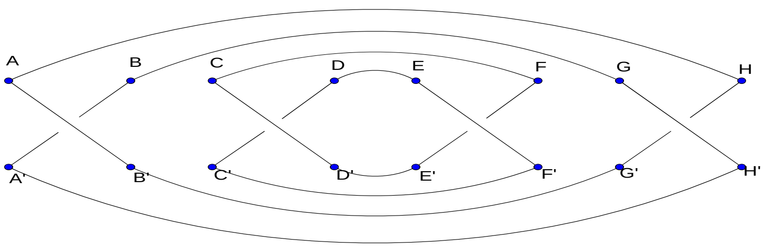


Fig. 11'

We cannot unite B with D neither with F because free ends would appear.

So ii) doesn't give any knot.

b) In two rows we have:

i) One row with three crosses and another row with one cross (See Fig. 12).

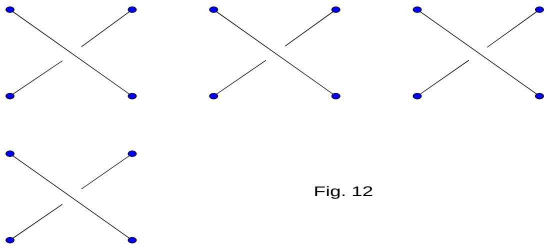


Fig. 12

In order not to undo crosses we have the two following possibilities:

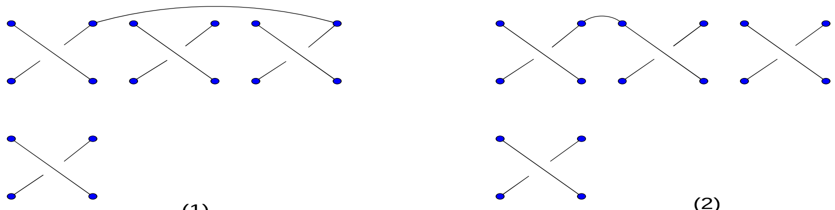


Fig. 13

Fig. 13.1 is not valid because it gives one free end.

Fig. 13.2 can be completed to Fig. 14.1 or fig. 14.2.

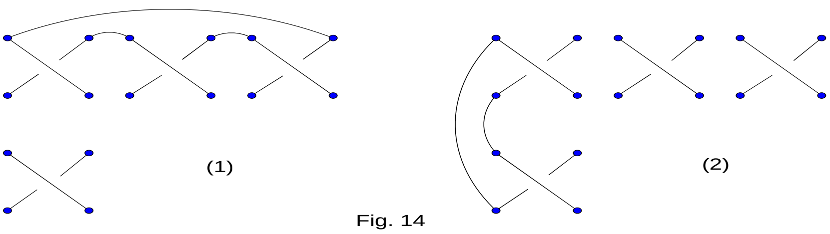


Fig. 14

Following fig. 14.1 we get

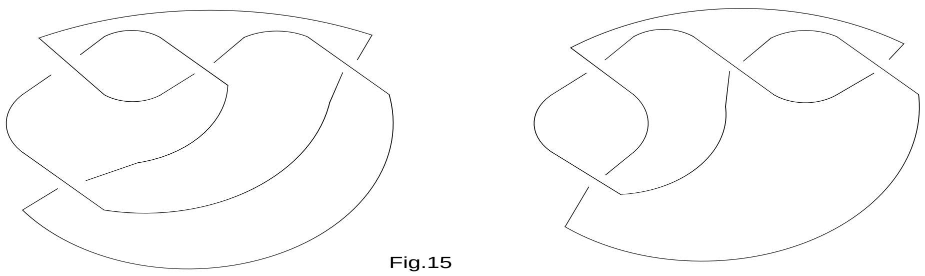


Fig.15

We have got the figure eight knot in both cases (See Fig. 15).

Changing the type of one croos point undoes the knot: if we change the type of the central cross we undo the knot (See Fig. 16). The same happens if we change the type of one of the other crosses.

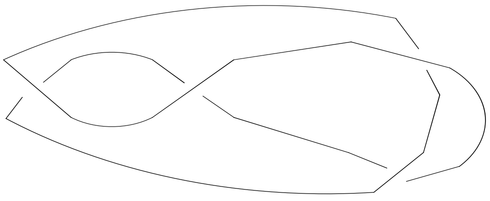


Fig.16

A simultaneous change in the type of the fist two crosses in the first row gives the trefoil knot (See Fig. 17). The same happens if we change simultaneously the type of the other two crosses.

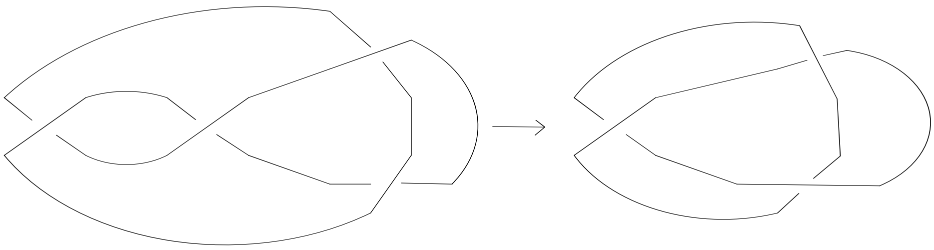


Fig.17

When we change three crosses we get the symmetrical respect to a mirror of the obtained changing only one cross, which are trivial.

When we change the type of the four crosses we obtain the symmetrical of the figure eight knot as we will see in Fig 18.

Amphichaerality of the figure eight knot:

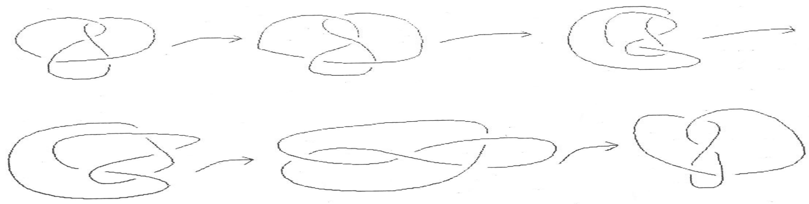


Fig. 18

Siguiendo con 14.2 tenemos:

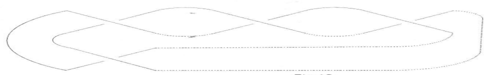


Fig.19

Following with figure 14.2 we have:

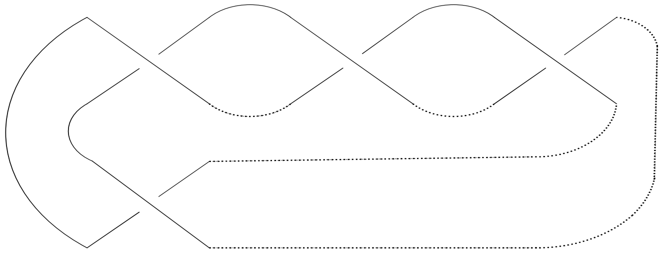


Fig.19

where the only possibility gives two connected components (See Fig. 19).

ii) Two rows with two crosses.

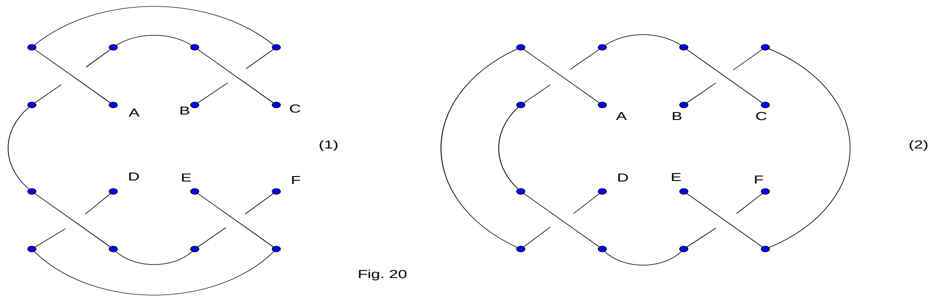


Fig. 20

If we unite in Fig. 20.1 C with F we obtain more than one connected component.

When we unite C with E, F remains free. When we unite C with D we have to unite E with F, then one cross in undone. When we unite C with A, B remains free. When we unite C with B one cross is undone. Therefore, Fig. 20.1 doesn't give any knot.

When we unite C with E, F remains free. When we unite C with D in Fig. 20.2 we have to unite E with F so that one cross is undone. When we unite C with A, B remains free. When we unite C with B one cross is undone. Therefore Fig. 20.2 doesn't give any knot.

c) The 1-manifolds obtained of three rows turns out to be the case of one column with three crosses and one column with one cross. This is the same as the case of one row with three crosses and another row with one cross by symmetry respect to te bisector of first and third square, already seen.

THE NON TRIVIAL KNOTS WITH FOUR DOUBLE POINTS ARE THE FIGURE EIGHT KNOT AND THE TREFOIL KNOT.

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